A Simple Extension of Fourier's Integral Theorem and some Physical Applications, in particular to the Theory of Quanta.

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(1) *Introductory.*—In Poincaré's* proof of the necessity of Planck's hypothesis of quanta, an essential stage of the argument† depends on the use of Fourier's integral theorem to invert a particular infinite integral. In the form used by Poincaré this theorem may be enunciated thus:—

Under suitable conditions, if

$$\Phi(\alpha) = \int_0^\infty e^{-\alpha\eta} w(\eta)\, d\eta, \tag{1}$$

then

$$w(\eta) = \frac{1}{2\pi i}\int_c \Phi(\alpha)\, e^{\alpha\eta}\, d\eta, \tag{2}$$

where C is a contour in the complex α-plane on which‡ $\mathbf{R}(\alpha) > \gamma > 0$ *and* $\mathbf{I}(\alpha)$ *goes from* $-i\infty$ *to* $+i\infty$.

Poincaré develops an argument which shows that, if $w(\eta)\,d\eta$ is the *a priori* probability that the energy of a resonating electron lies between η and $\eta + d\eta$, then $\Phi(\alpha)$ is such that $-d\{\log\Phi(\alpha)\}/d\alpha$ is the mean energy of the resonator at an absolute temperature C/α, where C is a known constant. When the mean energy of the resonator (of frequency ν) is known by experiment as a function of the absolute temperature, then $\Phi(\alpha)$ is known, except for an arbitrary constant multiplier. A direct appeal to formula (2) then shows that in these conditions, and with the same exception, $w(\eta)$ is also known and is, in fact, unique. It follows at once, and this is the object of Poincaré's work, that the known facts can be accounted for by one, and only one, function, $w(\eta)$—that is, in short, by the hypothesis of quanta.§

Unfortunately, the functions $w(\eta)$ and $\Phi(\alpha)$ that actually occur in the argument are such that Fourier's integral theorem does not apply at all. The actual value of $\Phi(\alpha)$ that turns up is $1/(1-e^{-\alpha\epsilon})$ and the integral (2) is meaningless. This leaves a gap in Poincaré's argument—a gap which it is desirable to fill. In order to do so, it is clear that we have to extend Fourier's integral theorem in such a way that an essentially discontinuous function such as the $w(\eta)$ of the last footnote is properly catered for. The type of extension required is to replace the integral (1) by what is really a Stieltjes' integral* of the form

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* 'Journal de Physique,' Ser. V, vol. 2, p. 5 (1912).

† *Loc. cit.*, Sec. 7.

‡ $\mathbf{R}(a)$ is the real part of a, $\mathbf{I}(a)$ is the real part of $-ia$.

§ Planck's (first) hypothesis of quanta is, of course, that the energy of the resonator can only be of the form $p\epsilon$, where p is a positive integer or zero, and $\epsilon = h\nu$, where h is Planck's constant. Thus Planck's hypothesis attributes the value zero to $w(\eta)$ except when $\eta = p\epsilon$, when $w(\eta)$ becomes infinite in such a way that $\int_{p\epsilon-\delta}^{p\epsilon+\delta} w(\eta)\,d\eta = 1$. Poincaré's thesis is that this is the only possible form of $w(\eta)$ which will lead to the correct $\Phi(a)$ fitting experimental facts.

$$\int_0^\infty e^{-\alpha\eta}\,d\omega(\eta),$$

where $\omega(\eta)$ is a monotonic (increasing) function of η, and to obtain the analogue of Fourier's integral theorem in this case.

I am not aware whether or no any such extension of Fourier's integral theorem is possible for the general Stieltjes' integral indicated above, but for the purpose of physical applications it is unnecessary to contemplate an integral so general. We proceed in the following section to make a limited extension of Fourier's integral theorem in this direction, which is yet sufficient to enable us to complete properly this gap in Poincaré's argument. There appears to be a possibility of further applications of this extension which is noted in the final sections.

(2) *A Generalisation of Fourier's Integral Theorem.*—For physical applications we may and shall confine ourselves to monotonic functions $\omega(\eta)$ which have a finite number of steps only in any finite range (η_0, η_1). In such a case we may suppose that the steps occur at the values η_i of η $(i = 0, 1, 2, \ldots, \eta, \ldots)$ where $\eta_i \longrightarrow \infty$ steadily as $i \longrightarrow \infty$, and that the size of the step at η_i is j_i. In such a case we need not really introduce the fundamental conception of a Stieltjes' integral at all. We may *define* generally the integral $\int f(\eta)d\omega(\eta)$, over any finite range in which $f(\eta)$ is continuous, to be the sum of the series $\Sigma f(\eta_i)j_i$ and the more ordinary integral $\int f(\eta)d\omega'(\eta)$, where $\omega'(\eta)$ is a *continuous* monotonic (increasing) function of η, which is simply $\omega(\eta)$ with the steps cut out. We shall still further limit the generality of the function $\omega(\eta)$ by supposing that $\omega'(\eta)$ has a definite differential coefficient $\Omega\,(\geqq 0)$ which exists and is continuous at all except a finite of points in any finite range (η_0, η_1), and finally that Ω is bounded in any (η_0, η_1). Thus the integral $\int f(\eta)\,d\omega'(\eta)$ is merely the ordinary Riemann integral $\int f(\eta)\,\Omega d\eta$. This definition is in agreement, in the restricted cases to which it applies, with the usual direct definition of a Stieltjes' integral. We can extend the definition, both converge. We can also extend the definition to include discontinuities

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* See *e.g.*, Hardy, 'Mess. Math.,' vol. 48, p. 90 (1919), where other references will be found.

to infinite ranges in the usual way provided that $\overset{\infty}{\Sigma} f(\eta_i) j_i$ and $\int^{\infty} f(\eta)\Omega d\eta$ and infinities of $f(\eta)$ provided that there are only a finite number of them in any finite range (η_0, η_1), that none of them coincides with any value of η_i, and, in the case of an infinity, that the corresponding part of the Riemann integral converges.

With these conventions as to the meaning of $\int f(\eta)\, d\omega(\eta)$ whenever such an integral occurs, we can proceed to our generalisation of Fourier's integral theorem. It will be well to enumerate precisely the conditions imposed by our conventions on $\omega(\eta)$. They are that *$\omega(\eta)$ is a monotonic (increasing) function of η for all values of η considered, which has only a finite number of simple discontinuities (steps) in any finite interval (η_0, η_1). The function $\omega'(\eta)$ which is derived from $\omega(\eta)$ by removal of the steps has a differential coefficient, Ω, which exists and is continuous except perhaps at a finite number of points in any finite range (η_0, η_1), and is also bounded in (η_0, η_1).* We shall refer to these conditions as conditions W. We can now enunciate and prove our extension of Fourier's integral theorem.

Theorem I.—The Extended Fourier's Integral Theorem.*

If $\omega(\eta)$ *satisfies conditions* W, *if* $\Sigma j_i e^{-\alpha\eta_i}$ *and* $\int^{\infty} e^{-\alpha\eta}\, \Omega\, d\eta$ *converge for* $\alpha = \gamma_0$, *and if*

$$\Phi(\alpha) = \int_{-a}^{\infty} e^{-\alpha\eta}\, d\omega(\eta), \tag{3}$$

then $\Phi(\alpha)$ *is a holomorphic function of* α *in the half plane* $\mathbf{R}(\alpha) > \gamma_0$, *and*

$$\tfrac{1}{2}\{\omega(\eta+0)+\omega(\eta-0)\} - \omega(-a) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Phi(\alpha)\, e^{\alpha\eta}\, \frac{d\alpha}{\alpha}. \tag{4}$$

where $\gamma > \gamma_0$, $\gamma > 0$, *and the infinite integral in* (4) *is to be evaluated as* $\underset{T\to\infty}{\mathrm{Lt}} \int_{\gamma-iT}^{\gamma+iT}$.

It is convenient in the enunciation of this theorem to use $-a$ and not zero as the lower limit of integration in (3), owing to the possibility of having the first step in $\omega(\eta)$ at $\eta = 0$. We can assume that a is so chosen that there is no step between $-a$ and 0.

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* [Added, June 28.—In the limited case which we consider here, this extension of Fourier's integral theorem is practically equivalent to a combination of Fourier's ordinary integral theorem (§ (1)) with Perron's formula for the sum of the first n coefficients of a Dirichlet's Series (Hardy and Riesz, 'Dirichlet's Series,' Cambridge Tracts No. 18, p. 12). It is simpler and more satisfactory to give a direct proof of this extension, but this essential equivalence should not be overlooked. I am indebted to the referees for this footnote.]

Under the conditions of the theorem $\sum_{i=0}^{\infty} j_i e^{-\alpha\eta_i}$ converges for $\alpha = \gamma_0$, and therefore represents a holomorphic function of α in the half plane $\mathbf{R}(\alpha) > \gamma_0$.* Under the same conditions $\int_{-a}^{\infty} e^{-\alpha\eta}\Omega d\eta$ also represents a holomorphic function of α in the same half plane.† Thus $\Phi(\alpha)$ is a holomorphic function of α in the region stated.

Now let

$$\begin{aligned} \mathrm{I}(\eta, \mathrm{T}) &= \frac{1}{2\pi i}\int_{\gamma-i\mathrm{T}}^{\gamma+i\mathrm{T}} \Phi(\alpha)\, e^{\alpha\eta}\frac{d\alpha}{\alpha}, \\ &= \frac{1}{2\pi i}\int_{\gamma-i\mathrm{T}}^{\gamma+i\mathrm{T}} e^{\alpha\eta}\frac{d\alpha}{\alpha}\int_{-a}^{\infty} e^{-\alpha\zeta}\, d\omega(\zeta), \qquad (\gamma > \gamma_0). \end{aligned}$$

As $\gamma > \gamma_0$, the inner integral is by definition the sum of a uniformly (and absolutely) convergent series and a uniformly and absolutely convergent Riemann integral for all values of α on the path of integration. We can, therefore, invert the order of the two integrations‡ and get

$$\mathrm{I}(\eta, \mathrm{T}) = \frac{1}{2\pi i}\int_{-a}^{\infty} d\omega(\zeta)\int_{\gamma-i\mathrm{T}}^{\gamma+i\mathrm{T}} e^{\alpha(\eta-\zeta)}\frac{d\alpha}{\alpha}. \tag{5}$$

Now we know§ that if $\gamma > 0, \mathrm{T} > 0$, and $y > 1$,

$$\left|\int_{\gamma-\mathrm{T}i}^{\gamma+\mathrm{T}i} y^s \frac{ds}{s} - 2\pi i\right| \leq \frac{2}{\mathrm{T}}\frac{y^\gamma}{\log y}, \tag{6}$$

while if $y < 1$,

$$\left|\int_{\gamma-\mathrm{T}i}^{\gamma+\mathrm{T}i} y^s \frac{ds}{s}\right| \leq \frac{2}{\mathrm{T}}\frac{y^\gamma}{\log(1/y)}, \tag{6'}$$

and if $y = 1$,

$$\left|\int_{\gamma-\mathrm{T}i}^{\gamma+\mathrm{T}i} \frac{ds}{s} - \pi i\right| \leq \frac{2\gamma}{\mathrm{T}}. \tag{6''}$$

In our case $y = e^{\eta-\zeta}$. We therefore substitute from equations (6) in equation (5) and obtain

$$\begin{aligned} \mathrm{I}(\eta, \mathrm{T}) &= \frac{1}{2\pi i}\left\{\int_{-a}^{\eta-\epsilon} + \int_{\eta-\epsilon}^{\eta+\epsilon} + \int_{\eta+\epsilon}^{\infty}\right\} d\omega(\zeta)\int_{\gamma-i\mathrm{T}}^{\gamma+i\mathrm{T}} e^{\alpha(\eta-\zeta)}\frac{d\alpha}{\alpha}, \\ &= \int_{-a}^{\eta-\epsilon} d\omega(\zeta) + \underline{\mathrm{O}}\left\{\frac{1}{\mathrm{T}}\int_{-a}^{\eta-\epsilon}\frac{e^{\gamma(\eta-\zeta)}}{\eta-\zeta}\, d\omega(\zeta)\right\} \\ &\quad + \underline{\mathrm{O}}\left\{\frac{1}{\mathrm{T}}\int_{\eta+\epsilon}^{\infty}\frac{e^{\gamma(\eta-\zeta)}}{\zeta-\eta}\, d\omega(\zeta)\right\} + \frac{1}{2\pi i}\int_{\eta-\epsilon}^{\eta+\epsilon} d\omega(\zeta)\int_{\gamma-i\mathrm{T}}^{\gamma+i\mathrm{T}} e^{\alpha(\eta-\zeta)}\frac{d\alpha}{\alpha}. \end{aligned}$$

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* Hardy and Riesz, *loc. cit.*, Theorem 4.

† Bromwich, 'Infinite Series,' p. 438.

‡ Bromwich, *loc. cit.*, pp. 117, 437.

§ Landau, 'Primzahlen,' vol. 1, p. 342.

Suppose η has any fixed value and that ϵ is chosen so small that the range $(\eta-\epsilon, \eta+\epsilon)$ contains no discontinuity of $\omega(\eta)$ other than (perhaps) η itself. Then

$$\mathrm{I}(\eta, \mathrm{T}) = \int_{-a}^{\eta-\epsilon} d\omega(\zeta) + \underline{O}\left(\frac{1}{\mathrm{T}\epsilon}\right) + \frac{1}{2\pi i}\int_{\eta-\epsilon}^{\eta+\epsilon} d\omega(\zeta) \int_{\gamma-i\mathrm{T}}^{\gamma+i\mathrm{T}} e^{\alpha(\eta-\zeta)}\frac{d\alpha}{\alpha}. \qquad (7)$$

It is easily verified that, when $-\epsilon \leq \eta-\zeta \leq \epsilon$ (T fixed),

$$\frac{1}{2\pi i}\int_{\gamma-i\mathrm{T}}^{\gamma+i\mathrm{T}} e^{\alpha(\eta-\zeta)}\frac{d\alpha}{\alpha} = \underline{O}(1),$$

and the integral is a continuous function of $\eta-\zeta$. Hence, if $\eta \neq \eta_i$, the last term in equation (7) is $\underline{O}(\epsilon)$ and we have

$$\mathrm{I}(\eta, \mathrm{T}) = \omega(\eta-\epsilon) - \omega(-a) + \underline{O}(1/\mathrm{T}\epsilon)\underline{O} + (\epsilon).$$

Now let $\epsilon = 1/\sqrt{\mathrm{T}}$, and let $\mathrm{T}\to\infty$. We find

$$\underset{\mathrm{T}\to\infty}{\mathrm{Lt}}\ \mathrm{I}(\eta, \mathrm{T}) = \omega(\eta-0) - \omega(-a) = \omega(\eta) - \omega(-a).$$

If however $\eta = \eta_i$, then by definition

$$\frac{1}{2\pi i}\int_{\eta-\epsilon}^{\eta+\epsilon} d\omega(\zeta)\int_{\gamma-i\mathrm{T}}^{\gamma+i\mathrm{T}} e^{\alpha(\eta-\zeta)}\frac{d\alpha}{\alpha} = \frac{1}{2\pi i} j_i\left[\pi i + \underline{O}(1/\mathrm{T})\right]$$

$$+\frac{1}{2\pi i}\int_{\eta-\epsilon}^{\eta+\epsilon} d\omega'(\zeta)\int_{\gamma-i\mathrm{T}}^{\gamma+i\mathrm{T}} e^{\alpha(\eta-\zeta)}\frac{d\alpha}{\alpha} = \tfrac{1}{2}j_i + \underline{O}(1/\mathrm{T}) + \underline{O}(\epsilon).$$

Thus in this case

$$\underset{\mathrm{T}\to\infty}{\mathrm{Lt}}\ \mathrm{I}(\eta, \mathrm{T}) = \omega(\eta-0) - \omega(-a) + \tfrac{1}{2}j_i = \tfrac{1}{2}\{\omega(\eta-0) + \omega(\eta+0)\} - \omega(-a).$$

The proof of Theorem I is completed.

(3) *The Extended Mellin's Integral Theorem.*—The connection between Mellin's and Fourier's integral inversion formulæ is well known—they are in fact almost the same theorem.* It is, therefore, to be expected that a generalisation of Mellin's formula is possible similar to the above for Fourier's, and this generalization is easily made. We have

Theorem II.—The Extended Mellin's Integral Theorem.

If $\omega(x)$ *satisfies conditions* W, *if* $\Sigma j_i x_i^{-s}$ *and* $\int^{\infty} x^{-s}\Omega\,dx$ *converge for* $s = \gamma_0$, *and if*

$$\Phi(s) = \int_a^{\infty} x^{-s}\,d\omega(x),$$

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* See *e.g.*, Hardy, 'Mess. Math.,' vol. 47, p. 178 (1918). [Added, June 28:— Formulæ (1) and (2) of this paper constitute Fourier's inversion formula (or integral theorem); the corresponding inversion formula of Mellin can be stated thus:—

Under suitable conditions, if

$$\phi(s) = \int_0^{\infty} w(x)\,x^{s-1}dx, \quad \text{then } w(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\phi(s)\,x^{-s}ds.$$

The connection between the two formulæ may be seen by substituting $x = e^{\eta}$.]

where $a > 0$ *and* x^{-s} *has its principal value* $\exp(-s\log x)$, *then* $\Phi(s)$ *is a holomorphic function of* s *in the half plane* $\mathbf{R}(s) > \gamma_0$, *and*

$$\tfrac{1}{2}\{\omega(x+0)+\omega(x-0)\}-\omega(a) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Phi(s)\,x^s\,\frac{ds}{s}, \tag{4'}$$

where $\gamma > \gamma_0$, $\gamma > 0$, *and the infinite integral in* (4′) *is to be evaluated as* $\underset{T\to\infty}{\mathrm{Lt}}\int_{\gamma-iT}^{\gamma+iT}$.

We omit the proof which is identical with that of Theorem I. It is easy to extend the theorem to the case of $\int_0^\infty x^{-s}\,d\omega(x)$ provided that suitable convergency conditions at the lower limit are satisfied.

(4) *Application to the Theory of Quanta.*—Poincaré's argument adapted to our notation shows that if Planck's law for the mean energy of a resonator of frequency ν is true, and if

$$\Phi(\alpha) = \int_{-a}^{\infty} e^{-\alpha\eta}\,d\omega(\eta),$$

where $\omega(\eta)$ is the *a priori* probability that the resonator possesses an energy at any moment between 0 and η (limits included), then

$$\Phi(\alpha) = 1/(1-\epsilon^{-\alpha\epsilon}), \qquad (\epsilon = h\nu). \tag{8}$$

Equation (8) may be taken here as being demanded by the experimental facts. If we insert this value of $\Phi(\alpha)$ in equation (4) and apply Theorem I we find that the only value of $\omega(\eta)$ of the form stipulated by conditions W which can produce this $\Phi(\alpha)$ is given by the equation

$$\tfrac{1}{2}\{\omega(\eta+0)+\omega(\eta-0)\}-\omega(-\alpha) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\alpha\eta}}{1-e^{-\alpha\epsilon}}\,\frac{d\alpha}{\alpha}. \tag{9}$$

Suppose that η has any fixed value which is not equal to $p\epsilon$, where p is a positive integer or zero. For definiteness we may suppose that $p\epsilon < \eta < (p+1)\epsilon$. On expanding the integrand we obtain

$$\begin{aligned}\tfrac{1}{2}\{\omega(\eta+0)+\omega(\eta-0)\}-\omega(-a) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\sum_{n=0}^{p} e^{\alpha(\eta-n\epsilon)}\,\frac{d\alpha}{\alpha}\\ &\quad+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{-\alpha[(p+1)\epsilon-\eta]}}{1-e^{-\alpha\epsilon}}\,\frac{d\alpha}{\alpha},\\ &= p+1+\mathrm{J}.\end{aligned}$$

J is the value of the last term on the right. But since $(p+1)\epsilon-\eta > 0$, it is easily proved that $\mathrm{J} = 0$ by applying Cauchy's theorem to the rectangle whose corners are $\gamma\pm i\mathrm{T}$, $\mathrm{X}\pm i\mathrm{T}$, and making $\mathrm{X}\longrightarrow\infty$, and $\mathrm{T}\longrightarrow\infty$. When

$\eta = p\epsilon$ the same reasoning applies except that the last term in the series contributes only $\frac{1}{2}$. Thus

$$\omega(\eta) - \omega(-a) = p+1, \qquad (p\epsilon < \eta < (p+1)\epsilon),$$

$$\tfrac{1}{2}\{\omega(\eta+0) + \omega(\eta-0)\} - \omega(-a) = p + \tfrac{1}{2}, \qquad (\eta = p\epsilon).$$

Now $\omega(\eta_1) - \omega(\eta_0)$ is the *a priori* probability* that the energy η of the resonator lies between η_0 and η_1. Hence this probability must be zero unless the range (η_0, η_1) includes one of the points $p\epsilon$, and unity when (η_0, η_1) includes any one such point. But this can only mean that the energy of the resonator may take any of the values $p\epsilon$ but no other values, and can therefore only change in either direction by jumps of amount ϵ, which is precisely Planck's (first) hypothesis. It hardly seems necessary to contemplate functions more general than those allowed by conditions W in a physical problem such as this.

In concluding this stage of the argument one should point out that, at one stage in Poincaré's proof, it is necessary to make use of the multiple integral.

$$\int_{-a}^{\infty} \cdots \int_{-a}^{\infty} e^{-\alpha\Sigma\eta}\, d\omega(\eta_1) \ldots d\omega(\eta_n)$$

which represents $[\Phi(\alpha)]^n$, and to show that it can be expressed in the form

$$\int_{-a}^{\infty} e^{-\alpha x}\, d\phi_n(x)\,;$$

this integral by Theorem I gives rise to the relation

$$\tfrac{1}{2}\{\phi_n(x+0) + \phi_n(x-0)\} - \phi_n(-a) = \frac{1}{2\pi i}\int_c [\Phi(\alpha)]^n e^{\alpha x} \frac{d\alpha}{\alpha}.$$

There is little difficulty in the necessary modification of Poincaré's work, but it must be given at length if at all, and I shall not consider the details here.

Since these sections were written, I have found that the same point is referred to by Planck in the Poincaré memorial volume of the 'Acta Mathematica' which has just appeared.† In section (3) of this article Planck criticises Poincaré's conclusion that his (Planck's) second hypothesis of quanta—discontinuous emission and continuous absorption—is inadmissible. He points out (p. 394) the gap in Poincaré's argument that I have attempted to fill up in this paper and the need for such a more complete discussion. But he suggests that it may be due to this gap in the mathematics that Poincaré's argument distinguishes between the first and second hypotheses, allowing the first to be necessary and the second impossible. I do not think that Planck's contention can be upheld, so far at least as it applies to the possibility of error in this gap. If the mean energy of an oscillator is to be

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* A constant multiplier omitted.

† 'Acta Mathematica,' vol. 38, p. 387 (1921).

$$\tfrac{1}{2}\epsilon+\frac{\epsilon}{e^{\alpha\epsilon}-1}$$

as Planck's second hypothesis requires, so that

$$\Phi(\alpha)=\frac{e^{-\frac{1}{2}\alpha\epsilon}}{1-e^{-\alpha\epsilon}},$$

and if the groundwork of Poincaré's argument is admitted to be correct, then it appears to follow inevitably that

$$\omega(\eta)=n, \qquad (n-\tfrac{1}{2})\epsilon<\eta<(n+\tfrac{1}{2})\epsilon.$$

But by the same reasoning that applies successfully to the former case, this must mean that the energy of the resonator can only be of the form $(n+\frac{1}{2})\epsilon$ and must therefore change discontinuously both in emission and absorption.

(5) *The Absorption of Non-homogeneous Beams of X-Rays.*—The foregoing ideas may possibly find another application in the discussion of absorption curves of non-homogeneous beams of X-rays, a question suggested to me by Mr. C. G. Darwin. Let μ be the absorption coefficient for rays of a definite wave-length, so that for such monochromatic rays

$$I(x)=I_0e^{-\mu x}. \tag{10}$$

In this equation I_0 is the original intensity of the beam, and $I(x)$ the intensity after it has penetrated a distance, x, through the given absorbing medium. The general radiation from an X-ray bulb consists of radiation with a continuous distribution of intensity over a wide range of wave-lengths (or values of μ), and superimposed on this the characteristic radiation (a line spectrum) of the anticathode if it is excited. In this line spectrum the intensity is concentrated at given wave-lengths, or given values of μ. If, therefore, we attempt to define a function $\phi(\mu)$ such that $\phi(\mu)\,d\mu$ is the intensity in the original beam of the radiation whose absorption coefficient lies between μ and $\mu+d\mu$, we are unable to do so, since $\phi(\mu)$ becomes infinite when we reach a line of the characteristic spectrum. This is in fact just the case we have had to consider in the theory of quanta. To include it, we make use of a function $\psi(\mu)$ which may be discontinuous and is such that $d\psi(\mu)$ is the intensity between μ and $\mu+d\mu$. The intensity $I(x)$ in the non-homogeneous beam after penetrating a thickness, x, will be given by the equation

$$I(x)=\int_0^\infty e^{-\mu x}\,d\psi(\mu), \tag{11}$$

which is an integral of just the form we have been discussing in this paper. We may assume that $\psi(\mu)$ satisfies conditions W. The problem before us is to determine $\psi(\mu)$, the intensity distribution from the known values of $I(x)$.

From the physical conditions of the problem $\int_0^\infty d\psi(\mu)$ must converge, for the intensity in the original beam is finite. The integral (11) therefore exists for all values of $\mathbf{R}(x) \geq 0$ and represents a holomorphic function of x in the right-hand half-plane. On applying Theorem I we should find at once that

$$\tfrac{1}{2}\{\psi(\mu+0)+\psi(\mu-0)\}-\psi(0) = \frac{1}{2\pi i}\int_c I(\alpha)\, e^{\alpha\mu}\,\frac{d\alpha}{\alpha},$$

where C is the contour $a-i\infty, a+i\infty, a > 0$. As it stands however this formula is probably of little or no practical value, for $I(x)$ is a function of x which is defined (observationally) only for real values of x. It will not in general be possible to deduce directly from the observed values of $I(x)$ its proper functional form or even an approximation to its proper functional form for general complex values of x. In fact it is only by a somewhat fortunate chance that this direct use of Theorem I can be made in the application to quanta. The function $\Phi(\alpha)$ there is really only determined (observationally) for real α, but it so happens that it has the precise functional form $1/(1-e^{-\alpha\epsilon})$ which defines it for complex α as well.

We can, however, turn the difficulty by the method applied by Schwartzschild* to a somewhat similar problem in the theory of stellar distributions. The method consists essentially in a use of Mellin's integral (Theorem II) to construct from $I(x)$ a function $J(q)$ which is known as a complex function of q—to which, therefore, Fourier's (or in this case Mellin's) integral formula can be applied.

Equation (11) defines $I(x)$. Let

$$J(q) = \int_0^\infty I(x)\, x^{q-1}\, dx, \qquad [\mathbf{R}(q) > 0]. \tag{12}$$

The integral converges absolutely at the lower limit if $\mathbf{R}(q) > 0$. It converges at the upper limit under the physical conditions of the problem, for in any beam of radiation there is a minimum wave-length—a minimum value μ_0 of μ—below which the intensity is necessarily zero. Thus, in effect

$$I(x) = \int_{\mu_0}^\infty e^{-\mu x}\, d\psi(\mu),$$

and it is evident that as $x \longrightarrow \infty$

$$I(x) = \underline{O}(e^{-\mu_0 x}).$$

We find therefore that

$$J(q) = \int_0^\infty x^{q-1}\, dx \int_{\mu_0}^\infty e^{-\mu x}\, d\psi(\mu).$$

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* Eddington, 'Stellar Motions and the Structure of the Universe,' p. 208.

This is an absolutely convergent repeated integral and the order of integration may, therefore, be inverted* provided that the inverted integral exists, as, in fact, it does here. This gives

$$J(q) = \int_{\mu_0}^{\infty} d\psi(\mu) \int_0^{\infty} e^{-\mu x} x^{q-1}\, dx,$$

or

$$\frac{J(q)}{\Gamma(q)} = \int_{\mu_0}^{\infty} \mu^{-q}\, d\psi(\mu), \qquad [\mathbf{R}(q) > 0]. \tag{13}$$

Applying Theorem II, we find at once that

$$\tfrac{1}{2}\{\psi(\mu+0)+\psi(\mu-0)\} = \frac{1}{2\pi i}\int_c \frac{J(q)}{\Gamma(q)} \mu^q \frac{dq}{q}, \qquad [\psi(\mu_0-0) = 0]. \tag{14}$$

Thus given the numerical values of the function $I(x)$ for real values of x, it is theoretically possible by two sets of integrations to determine the function $\psi(\mu)$ for all real values of μ. This method has actually been used in the problem of stellar distributions,† but it requires precise numerical data to make it practicable. Whether it could be applied to the analysis of actual X-ray absorption curves is open to question, and I do not propose to pursue the matter further here.

(6) *Summary.*—In this paper I have attempted to fill up a distinct gap in Poincaré's discussion of the necessity for quanta. It is at once clear that a mild form of Stieltjes' integral is a necessary instrument for this purpose; such integrals appear to occur naturally in other physical problems. The machinery of Fourier's integral theorem generalised in such a direction is thus of some importance; it is evident that the theorem must be true under much wider conditions than those under which I have proved it, and I recommend its more complete discussion to those who are interested in this branch of pure mathematics.

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* The inner integral is, of course, the sum of an absolutely and uniformly convergent series and an absolutely convergent Riemann integral.

† Eddington, *loc. cit.*, p. 220.